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Asymptotic expansion of a quadruple integral involving a Bessel function *

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Abstract: An asymptotic expansion is derived for a quadruple integral involving the Bessel function $J_0[\lambda(xy+zw)]$, where each of the integration variables x, y, z and w belongs to the interval [0, 1] and the large variable λ tends to infinity. Corresponding results are also obtained for similar integrals in two and three dimensions.

Keywords: Asymptotic expansion, quadruple integral, Bessel function, Hankel transform, crystallography.

1. Introduction

In a study of crystallography, Weiss [7] encountered the problem of finding an asymptotic expansion for the quadruple integral

$$Q(\lambda)=\int_0^{\pi/2}\cdots\int_0^{\pi/2}J_0(\lambda[\cos\alpha_1\cos\alpha_2+\cos\alpha_3\cos\alpha_4])\,\mathrm{d}\alpha_1\ldots\mathrm{d}\alpha_4, \tag{1.1}$$

where $J_0(x)$ is the Bessel function of the first kind of order zero. By using Neumann's addition theorem [4, p.59]

$$J_n(z_1+z_2)=\sum_{s=-\infty}^{\infty}J_s(z_1)J_{n-s}(z_2)$$

and the identity [2, p.47]

$$\frac{2}{\pi}\int_0^{\pi/2}J_l(\beta\cos\theta)\,\mathrm{d}\theta=J_{l/2}^2(\tfrac{1}{2}\beta),$$

he was able to show that

$$Q(\lambda)=(\tfrac{1}{2}\pi)^2\left[\varphi_0^2(\tfrac{1}{2}\lambda)+2\sum_{l=1}^{\infty}(-1)^l\varphi_{l/2}^2(\tfrac{1}{2}\lambda)\right], \tag{1.2}$$

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where

$$\varphi_\nu(\lambda)=\int_0^{\pi/2} J_\nu^2(\lambda\cos\theta)\,\mathrm{d}\theta. \tag{1.3}$$

An asymptotic expansion for the one-dimensional integral (1.3) has recently been derived; see [6,8]. In particular, it has been shown that

$$\varphi_\nu(\lambda)=\frac{1}{\pi\lambda}\left[\log\lambda+2\log 2-\psi\left(\tfrac{1}{2}+\nu\right)\right]+\mathrm{O}(\lambda^{-3/2}) \tag{1.4}$$

as $\lambda\to+\infty$. Since the coefficient of the leading term $(\log\lambda)/\lambda$ is independent of ν, it does not seem possible to obtain an asymptotic formula for $Q(\lambda)$ by simply substituting (1.4) into (1.2).

The purpose of this paper is to derive an asymptotic expansion for $Q(\lambda)$ directly from the quadruple integral (1.1). Our approach is to first reduce the integral (1.1) to a one-dimensional finite Hankel transform by using a change of variables, and then derive an asymptotic expansion for this finite transform. Our method actually applies to many similar types of integrals. For instance, we shall first consider the double integral

$$D(\lambda)=\int_0^1\int_0^1 J_\nu(\lambda xy)\frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\,\mathrm{d}x\,\mathrm{d}y,\quad \nu>-1. \tag{1.5}$$

Some of the results for this integral will be used in our construction of the asymptotic expansion for $Q(\lambda)$. By a simple change of variables, it is easily seen that the quantity inside the square bracket in (1.1) can be written in the form $xy+zw$; see (3.1). Both this and the argument xy of the Bessel function in (1.5) can be expressed as a difference of squares. This motivates us to include, as a further example, the derivation of an asymptotic expansion of the triple integral

$$T(\lambda)=\iiint_\Omega J_\nu(\lambda[z^2-x^2-y^2])\varphi(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad \nu>-1, \tag{1.6}$$

where φ is a C^∞-function of rapid decrease at infinity and Ω is the cone $\{(x,y,z)\colon x^2+y^2\leqslant z^2$ and $z\geqslant 0\}$.

2. The double integral $D(\lambda)$

In (1.5), we make the change of variables

$$t=xy\quad\text{and}\quad s=y. \tag{2.1}$$

Clearly

$$\frac{\partial(t,s)}{\partial(x,y)}=\begin{vmatrix} y & x\\ 0 & 1\end{vmatrix}=y, \tag{2.2}$$

and the double integral $D(\lambda)$ becomes

$$D(\lambda)=\int_0^1 h(t)J_\nu(\lambda t)\,\mathrm{d}t, \tag{2.3}$$

where

$$h(t)=\int_t^1\frac{1}{\sqrt{1-(t/s)^2}\sqrt{1-s^2}}\frac{\mathrm{d}s}{s}=\int_t^1\frac{1}{\sqrt{s^2-t^2}\sqrt{1-s^2}}\,\mathrm{d}s. \tag{2.4}$$

To obtain the asymptotic expansion of the finite Hankel transform in (2.3), we need the behavior of $h(t)$ near both endpoints $t=0$ and $t=1$. First note that upon making successive changes of variables $1-s^2=\sigma$ and $\sigma=(1-t^2)z$ in (2.4), we have

$$h(t)=\tfrac{1}{2}\int_0^1 \frac{1}{\sqrt{z(1-z)}\sqrt{1-(1-t^2)z}}\,\mathrm{d}z, \tag{2.5}$$

from which it is easily seen that $h(t)$ is an infinitely differentiable function in (0, 1] and the left-hand derivatives of $h(t)$ at $t=1$ of all orders can be readily calculated. Next we observe that the first integral in (2.4) is the Mellin convolution of the function

$$g(t)=\frac{1}{\sqrt{1-t^2}}\chi_{[0,1]}(t) \tag{2.6}$$

with itself, where $\chi_{[0,1]}(t)$ is the characteristic function of the interval [0, 1], i.e.,

$$h(t)=(g*g)(t)\equiv\int_0^\infty g(s)g(ts^{-1})s^{-1}\,\mathrm{d}s. \tag{2.7}$$

This observation motivates the proof of the following result.

Lemma 1. *As $t\to 0^+$, we have*

$$h(t)\sim\sum_{s=0}^{\infty}(a_s+b_s\log t)t^{2s}, \tag{2.8}$$

where

$$a_s=\frac{\pi}{(s!)^2\Gamma^2(-s+\frac{1}{2})}\{\psi(s+1)-\psi(-s+\tfrac{1}{2})\}, \tag{2.9}$$

ψ being the logarithmic Γ-function, and

$$b_s=-\frac{\pi}{(s!)^2\Gamma^2(-s+\frac{1}{2})}. \tag{2.10}$$

Proof. The Mellin transform of a locally integrable function on $(0,\infty)$ is defined by

$$M[f;z]=\int_0^\infty t^{z-1}f(t)\,\mathrm{d}t, \tag{2.11}$$

when the integral converges. Using the beta function integral, it is easy to show that

$$M[g;z]=\tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(\frac{1}{2}z)}{\Gamma[\frac{1}{2}(z+1)]}. \tag{2.12}$$

Since $h(t)=(g*g)(t)$, the Mellin transform of h is the square of the Mellin transform of g; see [9, p.151]. Thus

$$M[h;z]=\tfrac{1}{4}\pi\frac{\Gamma^2(\frac{1}{2}z)}{\Gamma^2[\frac{1}{2}(z+1)]}. \tag{2.13}$$

By the inversion formula [9, p.151],

$$h(t)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}t^{-z}\frac{\pi\Gamma^2(\frac{1}{2}z)}{4\Gamma^2[\frac{1}{2}(z+1)]}\,\mathrm{d}z, \tag{2.14}$$

where $c>0$. The integrand in (2.14) has poles at $z=0, -2, -4, \ldots$, and the vertical line of integration can be shifted to the left. From the Cauchy residue theorem, it follows that

$$h(t)=\sum_{s=0}^{N}\operatorname{Res}\left\{t^{-z}\frac{\pi\Gamma^2(\frac{1}{2}z)}{4\Gamma^2[\frac{1}{2}(z+1)]}: z=-2s\right\}+E(t), \tag{2.15}$$

where

$$E(t)=\frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}t^{-z}\frac{\pi\Gamma^2(\frac{1}{2}z)}{4\Gamma^2[\frac{1}{2}(z+1)]}\mathrm{d}z=\mathrm{o}(t^d) \tag{2.16}$$

as $t\to0^+$ and $2N<d<2N+2$. The o-estimate in (2.16) follows from a generalization of the Riemann–Lebesgue lemma given in [4, p.73]. We now recall the Laurent expansion [1, §1.17]

$$\Gamma(z)=\frac{(-1)^s}{s!}\left[\frac{1}{z+s}+\psi(s+1)+\mathrm{O}(z+s)\right], \tag{2.17}$$

from which it is easily calculated that

$$\operatorname{Res}\left\{t^{-z}\frac{\pi\Gamma^2(\frac{1}{2}z)}{4\Gamma^2[\frac{1}{2}(z+1)]}: z=-2s\right\}=(a_s+b_s\log t)t^{2s}, \tag{2.18}$$

where a_s and b_s are the constants given by (2.9) and (2.10). A combination of (2.15), (2.16) and (2.18) gives the required result (2.8). □

An alternative way of deriving the expansion (2.8) is to use the results in [10]. Since $h=g*g$, by Theorem 1 and Lemma 7 we have

$$h(t)=\sum_{s=0}^{n-1}(-1)^s\binom{-\frac{1}{2}}{s}\tau^{2s}*g+\sum_{s=0}^{n-1}(-1)^s\binom{-\frac{1}{2}}{s}g*\tau^{2s}+\mathrm{O}(t^{2n}\log t) \tag{2.19}$$

as $t\to0^+$, where

$$g*\tau^{2s}=A_{-2s}[g;-2s]t^{2s}, \tag{2.20}$$

$$\tau^{2s}*g=\left\{A_{-2s}[g;-2s]-(-1)^s\binom{-\frac{1}{2}}{s}\log t\right\}t^{2s}, \tag{2.21}$$

and

$$\begin{aligned}A_{-2s}[g;-2s]&=\lim_{z\to-2s}\left\{M[g;z]-\frac{\sqrt{\pi}}{2}\frac{(-1)^s}{s!}\frac{2}{\Gamma(-s+\frac{1}{2})}\frac{1}{z+2s}\right\}\\&=\frac{\sqrt{\pi}}{2}\frac{(-1)^s}{s!}\frac{1}{\Gamma(-s+\frac{1}{2})}[\psi(s+1)-\psi(-s+\tfrac{1}{2})];\end{aligned} \tag{2.22}$$

cf. (2.12) and (2.17). For the order estimate in (2.19), see also the explanation given in [10, p.947]. Expansion (2.8) now follows from substituting (2.20)–(2.22) in (2.19).

We are now ready to state a result of Soni [5], which is immediately applicable to the integral (2.3). Let $f(t)$ be an infinitely differentiable function in the interval $0<t\leqslant a$ with an asymptotic expansion of the form

$$f(t)\sim\sum_{s=0}^{\infty}t^{\alpha_s}P_s(\log t), \quad t\to0^+, \tag{2.23}$$

where $P_s(w)$ is a polynomial of degree $m = m(s)$, and $\{\alpha_s\}$ is an increasing sequence tending to infinity. For every $n \geqslant 1$, we write

$$f(t) = \sum_{s=0}^{n-1} t^{\alpha_s} P_s(\log t) + f_n(t),$$

and assume that

$$f_n^{(j)}(t) = \mathrm{O}\left\{t^{\alpha_n - j}(\log t)^{m(n)}\right\}, \quad t \to 0^+,$$

for every $j \geqslant 0$. Let p be an arbitrary fixed positive integer, and let n be the largest nonnegative integer such that

$$\alpha_{n-1} < p + \tfrac{1}{2}. \tag{2.24}$$

Soni's result states that

$$\int_0^a f(t) J_\nu(xt)\, \mathrm{d}t = \sum_{s=0}^{n-1} P_s(D_s)\left[\Lambda(\alpha_s) x^{-\alpha_s - 1}\right] + \sum_{k=1}^{p} c_k J_{\nu+k}(xa) x^{-k} + \mathrm{o}(x^{-p-\frac{1}{2}}) \tag{2.25}$$

provided that $\alpha_0 + \nu > -1$, where

$$\Lambda(\alpha_s) = 2^{\alpha_s} \frac{\Gamma\left(\frac{1}{2}\alpha_s + \frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\alpha_s + \frac{1}{2}\right)}, \tag{2.26}$$

$$c_k = (-1)^{k+1} a^{\nu+k}\left[\mathscr{D}^{k-1} t^{-\nu-1} f(t)\right]_{t=a}, \tag{2.27}$$

and D_s and $\mathscr{D}$ are the differential operators defined by

$$D_s = \frac{\mathrm{d}}{\mathrm{d}\alpha_s} \quad \text{and} \quad \mathscr{D} = t^{-1}\frac{\mathrm{d}}{\mathrm{d}t},$$

respectively. It is easy to show that

$$\begin{aligned} D_s\left[\Lambda(\alpha_s) x^{-\alpha_s-1}\right] = \Lambda(\alpha_s)\Big[\log 2 + \tfrac{1}{2}\psi\left(\tfrac{1}{2}\alpha_s + \tfrac{1}{2}\nu + \tfrac{1}{2}\right) + \tfrac{1}{2}\psi\left(\tfrac{1}{2}\nu - \tfrac{1}{2}\alpha_s + \tfrac{1}{2}\right) \\ - \log x\Big] x^{-\alpha_s - 1} \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} &D_s^2\left[\Lambda(\alpha_s) x^{-\alpha_s-1}\right] \\ &\quad = \Lambda(\alpha_s)\Big\{\left[\log 2 + \tfrac{1}{2}\psi\left(\tfrac{1}{2}\alpha_s + \tfrac{1}{2}\nu + \tfrac{1}{2}\right) + \tfrac{1}{2}\psi\left(\tfrac{1}{2}\nu - \tfrac{1}{2}\alpha_s + \tfrac{1}{2}\right)\right]^2 \\ &\qquad + \left[\tfrac{1}{4}\psi'\left(\tfrac{1}{2}\alpha_s + \tfrac{1}{2}\nu + \tfrac{1}{2}\right) - \tfrac{1}{4}\psi'\left(\tfrac{1}{2}\nu - \tfrac{1}{2}\alpha_s + \tfrac{1}{2}\right)\right] \\ &\qquad - 2\left[\log 2 + \tfrac{1}{2}\psi\left(\tfrac{1}{2}\alpha_s + \tfrac{1}{2}\nu + \tfrac{1}{2}\right) + \tfrac{1}{2}\psi\left(\tfrac{1}{2}\nu - \tfrac{1}{2}\alpha_s + \tfrac{1}{2}\right)\right] \log x \\ &\qquad + (\log x)^2\Big\} x^{-\alpha_s - 1}. \end{aligned} \tag{2.29}$$

If the upper limit in (2.25) is infinite and if

$$f^{(j)}(t) = \mathrm{O}(t^{-\frac{1}{2}-j-\epsilon}), \quad t \to \infty, \tag{2.30}$$

for every $j=0, 1, \ldots$ and for some $\varepsilon>0$, then the second series in (2.25) vanishes and we have

$$\int_0^\infty f(t)J_\nu(xt)\,\mathrm{d}t=\sum_{s=0}^{n-1}P(D_s)\left[\Lambda(\alpha_s)x^{-\alpha_s-1}\right]+\mathrm{o}(x^{-p-\frac{1}{2}}), \tag{2.31}$$

as $x\to\infty$, where p and n satisfy the relation (2.24). The last result can of course be rewritten as

$$\int_0^\infty f(t)J_\nu(xt)\,\mathrm{d}t=\sum_{s=0}^{N-1}P(D_s)\left[\Lambda(\alpha_s)x^{-\alpha_s-1}\right]+\mathrm{O}\left\{x^{-\alpha_N-1}(\log x)^{m(N)}\right\}, \tag{2.32}$$

as $x\to\infty$, for every $N\geqslant 1$.

In the case of (2.3), we have $a=1$, $x=\lambda$ and $f(t)=h(t)$, where $h(t)$ is given by (2.8). Thus, $\alpha_s=2s$ and $P_s(w)=a_s+b_sw$ for $s=0, 1, 2, \ldots$. From (2.5) and (2.27), it is easily computed that $c_1=\frac{1}{2}\pi$. Since $a_0=2\log 2$ and $b_0=-1$, coupling (2.25) and (2.28) gives

$$D(\lambda)=\left[\log 2-\psi\left(\tfrac{1}{2}\nu+\tfrac{1}{2}\right)+\log\lambda\right]\lambda^{-1}+\tfrac{1}{2}\pi J_1(\lambda)\lambda^{-1}+\mathrm{o}(\lambda^{-3/2}) \tag{2.33}$$

as $\lambda\to\infty$.

3. The quadruple integral $Q(\lambda)$

In (1.1), we first make the change of variables $x=\cos\alpha_1$, $y=\cos\alpha_2$, $z=\cos\alpha_3$ and $w=\cos\alpha_4$. This yields

$$Q(\lambda)=\int_0^1\cdots\int_0^1 J_0[\lambda(xy+zw)]\frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}\sqrt{1-w^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w. \tag{3.1}$$

Motivated by (2.1), we make the further change of variables

$$s=xy+zw,\qquad t=xy,\qquad u=x,\qquad v=w. \tag{3.2}$$

Note that for (x, y, z, w) in the unit cube, s runs between 0 and 2, while t runs between 0 and s if $0\leqslant s\leqslant 1$, or, since $zw\leqslant 1$, between $s-1$ and 1 if $s>1$. Since $t=xy=uy$ and $y\leqslant 1$, for fixed t we have u running between t and 1. Similarly, since $s-t=zw=zv$ and $z\leqslant 1$, for fixed s and t we have v running between $s-t$ and 1. From (3.2), it is also easily computed that

$$\frac{\partial(s, t, u, v)}{\partial(x, y, z, w)}=-xw=-uv. \tag{3.3}$$

Hence, we obtain

$$\begin{aligned}Q(\lambda)=&\int_0^1\int_0^s\int_t^1\int_{s-t}^1 J_0(\lambda s)\frac{\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}t\,\mathrm{d}s}{\left(\sqrt{1-u^2}\sqrt{1-(t/u)^2}\sqrt{1-[(s-t)/v]^2}\sqrt{1-v^2}\right)uv}\\&+\int_1^2\int_{s-1}^1\int_t^1\int_{s-t}^1 J_0(\lambda s)\frac{\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}t\,\mathrm{d}s}{\left(\sqrt{1-u^2}\sqrt{1-(t/u)^2}\sqrt{1-[(s-t)/v]^2}\sqrt{1-v^2}\right)uv}.\end{aligned} \tag{3.4}$$

The iterated integral with respect to $\mathrm{d}v\,\mathrm{d}u$ can be written as

$$\left(\int_t^1 \frac{\mathrm{d}u}{\left(\sqrt{1-u^2}\sqrt{1-(t/u)^2}\right)u}\right)\left(\int_{s-t}^1 \frac{\mathrm{d}v}{\left(\sqrt{1-v^2}\sqrt{1-[(s-t)/v]^2}\right)v}\right).$$

The first factor of this product is simply the function $h(t)$ defined in (2.4), and the second factor is $h(s-t)$. Consequently, (3.4) is reduced to

$$Q(\lambda)=\int_0^1 J_0(\lambda s)\int_0^s h(t)h(s-t)\,\mathrm{d}t\,\mathrm{d}s+\int_1^2 J_0(\lambda s)\int_{s-1}^1 h(t)h(s-t)\,\mathrm{d}t\,\mathrm{d}s. \tag{3.5}$$

Since $h(t)=0$ for $t>1$ in our case, the last two integrals can be combined, and we get

$$Q(\lambda)=\int_0^2 J_0(\lambda s)\int_0^s h(t)h(s-t)\,\mathrm{d}t\,\mathrm{d}s. \tag{3.6}$$

We shall define

$$k(t)=\int_0^t h(\tau)h(t-\tau)\,\mathrm{d}\tau, \tag{3.7}$$

so that the iterated integral in (3.6) becomes the one-dimensional finite Hankel transform

$$Q(\lambda)=\int_0^2 k(t)J_0(\lambda t)\,\mathrm{d}t. \tag{3.8}$$

We now proceed to derive the asymptotic expansion of $k(t)$ as $t\to 0^+$. First we recall that the convolution of two locally integrable functions f and g on $[0,\infty)$ is defined by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau. \tag{3.9}$$

The convolution defined in (3.9), and used in (3.10)–(3.13), arises naturally in the Laplace transform theory *and differs from that defined in* (2.7). In terms of this convolution, (3.7) can be written as

$$k(t)=(h*h)(t). \tag{3.10}$$

Using the beta integral formula [1, p.9], it is easily seen that

$$t^{\mu}*t^{\nu}=\frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}t^{\mu+\nu+1} \tag{3.11}$$

whenever $\mu>-1$ and $\nu>-1$. Upon differentiation with respect to μ, (3.11) gives

$$t^{\mu}(\log t)*t^{\nu}=\frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}\{[\psi(\mu+1)-\psi(\mu+\nu+2)]+\log t\}t^{\mu+\nu+1}. \tag{3.12}$$

A corresponding formula results for $t^{\mu}*t^{\nu}(\log t)$. By the same reasoning, we also have

$$t^{\mu}(\log t)*t^{\nu}(\log t)=\frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}\left[a(\mu,\nu)+b(\mu,\nu)\log t+\log^2 t\right]t^{\mu+\nu+1}, \tag{3.13}$$

where

$$a(\mu, \nu) = [\psi(\mu+1) - \psi(\mu+\nu+2)][\psi(\nu+1) - \psi(\mu+\nu+2)] - \psi'(\mu+\nu+2) \tag{3.14}$$

and

$$b(\mu, \nu) = \psi(\mu+1) + \psi(\nu+1) - 2\psi(\mu+\nu+2). \tag{3.15}$$

Substituting (2.8) in (3.10) and carrying out the convolutions term-by-term, it is easily shown that

$$k(t) \sim \sum_{s=0}^{\infty} (A_s + B_s \log t + C_s \log^2 t) t^{2s+1}, \tag{3.16}$$

as $t \to 0^+$, where the coefficients A_s, B_s and C_s can all be expressed explicitly in terms of the Γ- and the ψ-functions. The first set of coefficients is given by

$$A_0 = (2 \log 2)^2 + 4 \log 2 + (2 - \tfrac{1}{3}\pi^2), \qquad B_0 = -4 \log 2 - 2, \qquad C_0 = 1. \tag{3.17}$$

To the integral (3.8), we can now apply the result in (2.25) with $a = 2$, $x = \lambda$, $\nu = 0$ and $f(t)$ replaced by $k(t)$. Thus, in (2.23), we have $\alpha_s = 2s + 1$ and $P_s(w) = A_s + B_s w + C_s w^2$. Recall that

$$\frac{1}{\Gamma(z)} = (-1)^m m!(z+m) + \text{higher-order terms} \tag{3.18}$$

and

$$\psi(z) = -\frac{1}{z+m} + \psi(m+1) + \text{higher-order terms}. \tag{3.19}$$

The first equation is obtained by expanding $\sin \pi z$ and $\Gamma(1-z)$ in the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ into Taylor series near $z = -m$. The second equation is the Laurent expansion of $\psi(z)$ given in [1, p.47, equation (12)]. From (3.18) and (3.19), it follows that

$$\left.\frac{1}{\Gamma(z)}\right|_{z=-m} = 0, \qquad \frac{1}{\Gamma(z)}\psi(z)|_{z=-m} = (-1)^{m+1} m!,$$

$$\psi'(z) = \frac{1}{(z+m)^2} + \text{Constant} + o(1)$$

and

$$\frac{1}{\Gamma(z)}[\psi^2(z) - \psi'(z)]|_{z=-m} = 2(-1)^{m+1}\psi(m+1)m!.$$

Consequently, we obtain from (2.26), (2.28) and (2.29)

$$\Lambda(\alpha_s)\lambda^{-\alpha_s-1}|_{\alpha_s=2s+1} = 0, \qquad D_s[\Lambda(\alpha_s)\lambda^{-\alpha_s-1}]|_{\alpha_s=2s+1} = (-1)^{s+1} 2^{2s}(s!)^2 \lambda^{-2s-2},$$

and

$$D_s^2[\Lambda(\alpha_s)\lambda^{-\alpha_s-1}]|_{\alpha_s=2s+1} = (-1)^{s+1} 2^{2s+1}(s!)^2[\log 2 + \psi(s+1) - \log \lambda]\lambda^{-2s-2}$$

for $s = 0, 1, 2, \ldots$. Soni's result in (2.25) then gives

$$Q(\lambda) = -[B_0 + 2C_0(\log 2 - \gamma) - 2C_0 \log \lambda]\lambda^{-2} + c_1 J_1(2\lambda)\lambda^{-1} + c_2 J_2(2\lambda)\lambda^{-2} + o(\lambda^{-2-1/2}), \tag{3.20}$$

where γ is the Euler–Mascheroni constant and B_0, C_0 are given explicitly in (3.17). The coefficients c_1 and c_2 are defined by (2.27) with $\nu = 0$, $a = 2$ and $f(t)$ replaced by $k(t)$. Since $h(t) = 0$ for $t > 1$, (3.7) becomes

$$k(t) = \int_{t-1}^{1} h(\tau)h(t-\tau)\,\mathrm{d}\tau$$

for t near 2, from which it follows that

$$\lim_{t\to 2^-} k(t) = 0.$$

By Leibniz's rule,

$$k'(t) = \int_{t-1}^{1} h(\tau)h'(t-\tau)\,\mathrm{d}\tau - h(t-1)h(1).$$

Thus, we also have

$$\lim_{t\to 2^-} k'(t) = -h^2(1) = -\tfrac{1}{4}\pi^2.$$

Consequently

$$c_1 = 0, \qquad c_2 = \tfrac{1}{4}\pi^2. \tag{3.21}$$

Inserting (3.17) and (3.21) in (3.20) gives

$$Q(\lambda) = 2[\log 2 + (1+\gamma) + \log\lambda]\lambda^{-2} + \tfrac{1}{4}\pi^2 J_2(2\lambda)\lambda^{-2} + o(\lambda^{-2-1/2}), \tag{3.22}$$

which is quite different from the asymptotic formula that one would obtain from the first term in (1.2) by using (1.4).

4. The triple integral $T(\lambda)$

By using the method of resolution of multiple integrals [9, p.280], the integral (1.6) can be reduced to the one-dimensional Hankel transform

$$T(\lambda) = \int_0^\infty f(t)J_\nu(\lambda t)\,\mathrm{d}t, \tag{4.1}$$

where $f(t)$ is the surface integral

$$f(t) = \iint_{\Gamma_t} \frac{\varphi(x, y, z)}{|\nabla P|}\,\mathrm{d}S \tag{4.2}$$

with $P(x, y, z) = z^2 - x^2 - y^2$, $\Gamma_t = \{(x, y, z)\colon P(x, y, z) = t,\ z > 0\}$ and $\mathrm{d}S$ being the surface element of the hyperboloid $z^2 - x^2 - y^2 = t$. Using polar coordinates, we can rewrite (4.2) more explicitly as

$$f(t) = \tfrac{1}{2}\int_0^\infty \int_0^{2\pi} \varphi\left(r\cos\theta,\ r\sin\theta,\ \sqrt{r^2+t}\right)\frac{r}{\sqrt{r^2+t}}\,\mathrm{d}\theta\,\mathrm{d}r. \tag{4.3}$$

Since φ is rapidly decreasing [9, p.266], $f(t)$ satisfies the condition given in (2.30).

To derive the asymptotic expansion of $f(t)$ as $t \to 0^+$, one may attempt to proceed directly from (4.3). However, one will quickly find out that this procedure encounters formidable difficulty. Therefore we proceed in a different direction. Consider the Mellin transform

$$M[f;s] = \int_0^\infty t^{s-1} f(t)\,\mathrm{d}t, \tag{4.4}$$

which is clearly analytic in the half-plane Re $s > 0$. Again by the method of resolution of multiple integrals, (4.4) can be written as

$$M[f;s] = \iiint_\Omega (z^2 - x^2 - y^2)^{s-1} \varphi(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{4.5}$$

where Ω is the cone $\{(x, y, z){:}\ z^2 - x^2 - y^2 \geqslant 0 \text{ and } z \geqslant 0\}$. If the integration domain in (4.5) is the entire region $z^2 - x^2 - y^2 \geqslant 0$, instead of just the upper half, then (4.5) can be viewed as the action of the generalized function P_+^λ on the test function φ, with $\lambda = s - 1$ and P being the quadratic form $z^2 - x^2 - y^2$; see [3, p.253]. In what follows, we shall show that the Mellin transform $M[f;s]$ can be analytically continued to a meromorphic function with simple poles at $s = 0, -1, -2, \ldots$ and $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. Our method is a modification of that given in [3, Chapter III, §2.2]. We include it here for three purposes: (i) to justify some of the incomplete arguments given in the above reference, (ii) to avoid the concepts of regularization of divergent integrals and delta functions associated with quadratic forms, and, most importantly (iii) to correct an error in that reference which is not easily detected (see the remark following Lemma 2 below).

Returning to the integral in (4.5), we first introduce the cylindrical coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

which gives

$$M[f;s] = \int_0^\infty \int_0^z (z^2 - r^2)^{s-1} \psi(z, r) r\,\mathrm{d}r\,\mathrm{d}z, \tag{4.6}$$

with

$$\psi(z, r) = \int_0^{2\pi} \varphi(r\cos\theta, r\sin\theta, z)\,\mathrm{d}\theta. \tag{4.7}$$

Next we make the change of variables $u = z^2$ and $v = r^2$ so that (4.6) becomes

$$M[f;s] = \tfrac{1}{4}\int_0^\infty \int_0^u (u - v)^{s-1} \psi_1(u, v) u^{-1/2}\,\mathrm{d}v\,\mathrm{d}u, \tag{4.8}$$

where

$$\psi_1(u, v) = \psi(\sqrt{u}, \sqrt{v}) = \psi(z, r). \tag{4.9}$$

Finally we make the substitution $v = ut$. This yields

$$M[f;s] = \tfrac{1}{4}\int_0^\infty u^{s-1/2} \int_0^1 (1 - t)^{s-1} \psi_1(u, ut)\,\mathrm{d}t\,\mathrm{d}u. \tag{4.10}$$

Set

$$\Phi(s, u) = \tfrac{1}{4}\int_0^1 (1 - t)^{s-1} \psi_1(u, ut)\,\mathrm{d}t. \tag{4.11}$$

Clearly, $\Phi(s, u)$ is analytic in s for Re $s > 0$. Coupling (4.10) and (4.11), we have

$$M[f;s] = \int_0^{\infty} u^{s-1/2}\Phi(s, u)\,\mathrm{d}u. \tag{4.12}$$

The function $\psi_1(u, ut)$ can be expanded into a finite Taylor series at $t = 1$ plus a remainder. Thus

$$\psi_1(u, ut) = \sum_{k=0}^{N} (-1)^k a_k(u)(1-t)^k + R_N(u, t) \tag{4.13}$$

for every nonnegative integer N, where

$$a_k(u) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\psi_1(u, ut)\right]_{t=1} \tag{4.14}$$

and

$$R_N(u, t) = \mathrm{O}\{(1-t)^{N+1}\} \quad \text{as } t \to 1^-. \tag{4.15}$$

Since φ is rapidly decreasing, the O-symbol holds uniformly for all u. Inserting (4.13) in (4.11), and integrating term-by-term gives

$$\Phi(s, u) = \sum_{k=0}^{N} \frac{(-1)^k}{4}\frac{a_k(u)}{s+k} + \tfrac{1}{4}\int_0^1 (1-t)^{s-1} R_N(u, t)\,\mathrm{d}t. \tag{4.16}$$

The last term in (4.16) is analytic for Re $s > -N-1$. Consequently, this equation extends $\Phi(s, u)$ to a meromorphic function in Re $s > -N-1$ with simple poles at $s = 0, -1, -2, \ldots, -N$. Since N is arbitrary, we conclude that $\Phi(s, u)$ can be extended to a meromorphic function in the entire plane with simple poles at $s = 0, -1, -2, \ldots, -k, \ldots$. From (4.16), we also have

$$\operatorname*{Res}_{s=-k} \Phi(s, u) = \frac{(-1)^k}{4} a_k(u) = \frac{(-1)^k}{4(k!)}\left[\frac{\partial^k}{\partial t^k}\psi_1(u, ut)\right]_{t=1}. \tag{4.17}$$

We now examine the behavior of $\psi_1(u, ut)$ near $u = 0$. From (4.7) and (4.9), it follows that

$$\psi_1(u, ut) = \int_0^{2\pi} \varphi(\sqrt{u}\sqrt{t}\cos\theta, \sqrt{u}\sqrt{t}\sin\theta, \sqrt{u})\,\mathrm{d}\theta. \tag{4.18}$$

By Taylor's theorem,

$$\varphi(x, y, z) = \sum_{0 \leqslant i+j+k \leqslant N} c_{ijk} x^i y^j z^k + E_N(x, y, z) \tag{4.19}$$

for every nonnegative integer N, where

$$c_{ijk} = \frac{1}{i!j!k!}\frac{\partial^{i+j+k}\varphi}{\partial x^i \partial y^j \partial z^k}(0, 0, 0) \tag{4.20}$$

and

$$E_N(x, y, z) = \sum_{i+j+k=N+1} \frac{x^i y^j z^k}{i!j!k!}\frac{\partial^{i+j+k}\varphi}{\partial x^i \partial y^j \partial z^k}(\xi, \eta, \zeta), \tag{4.21}$$

(ξ, η, ζ) being a point on the line segment joining $(0, 0, 0)$ to (x, y, z). Since

$$\int_0^{2\pi} \cos^i\theta \sin^j\theta\,\mathrm{d}\theta = 0$$

when either i or j or both are odd, and since

$$\int_0^{2\pi} \cos^{2m}\theta \, \sin^{2n}\theta \, d\theta = 2\frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(m+n)!},$$

termwise integration gives

$$\psi_1(u, ut) = \sum_{l=0}^{N} \alpha_l(t) u^{l/2} + O\{u^{(N+1)/2}\} \quad \text{as } u \to 0^+, \tag{4.22}$$

where the O-term is uniform with respect to t in [0, 1] and

$$\alpha_l(t) = 2 \sum_{2m+2n+k=l} c_{2m,2n,k} \frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(m+n)!} t^{m+n}. \tag{4.23}$$

Furthermore, we have from (4.11)

$$\Phi(s, u) = \sum_{l=0}^{N} \beta_l(s) u^{l/2} + \Psi_N(s, u), \tag{4.24}$$

where

$$\beta_l(s) = \sum_{2m+2n+k=l} \frac{c_{2m,2n,k}}{s(s+1)\cdots(s+m+n)} \frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{2} \tag{4.25}$$

and the remainder $\Psi_N(s, u)$ is a meromorphic function in the entire plane with simple poles at $s = 0, -1, -2, \ldots$ and is $O\{u^{(N+1)/2}\}$ for small values of u.

Returning to (4.12), we write

$$M[f;s] = \int_0^1 u^{s-1/2}\Phi(s, u) \, du + \int_1^{\infty} u^{s-1/2}\Phi(s, u) \, du. \tag{4.26}$$

The second integral on the right-hand side exists for all values of s except $s = 0, -1, -2, \ldots$, and is a meromorphic function in the plane with simple poles at these points. In the first integral in (4.26), we replace $\Phi(s, u)$ by its expansion (4.24). This gives

$$\int_0^1 u^{s-1/2}\Phi(s, u) \, du = \sum_{l=0}^{N} \beta_l(s) \frac{1}{s+\frac{1}{2}+\frac{1}{2}l} + \int_0^1 u^{s-1/2}\Psi_N(s, u) \, du. \tag{4.27}$$

The last integral exists for Re $s > -\frac{1}{2}N - 1$. The terms under the summation sign are meromorphic functions with finite number of *simple* poles at zero, negative integers and negative half-integers; see (4.25). Since N is arbitrary, equation (4.27) extends the function defined for Re $s > -\frac{1}{2}$ by the first integral on the right-hand side of (4.26). Thus, the Mellin transform $M[f;s]$ is extended to the entire plane.

The residue of $M[f;s]$ at half integers, say $s = -p - \frac{1}{2}$, can be easily calculated from (4.26) and (4.27), and we have

$$\operatorname*{Res}_{s=-p-\frac{1}{2}} M[f;s] = \beta_{2p}(s)\big|_{s=-p-1/2}. \tag{4.28}$$

From (4.25), it follows that

$$\operatorname*{Res}_{s=-p-1/2} M[f;s] = \sum_{m+n+l=p} \frac{(-1)^{m+n+1} c_{2m,2n,2l}}{(p+\frac{1}{2})(p-\frac{1}{2})\cdots(p+\frac{1}{2}-m-n)} \times \frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{2}. \tag{4.29}$$

Putting $p=0$ and $p=1$, we obtain from (4.20)

$$\operatorname*{Res}_{s=-1/2} M[f;s] = -\pi\varphi(0,0,0) \tag{4.30}$$

and

$$\operatorname*{Res}_{s=-3/2} M[f;s] = \tfrac{1}{6}\pi\{\varphi_{xx}+\varphi_{yy}-\varphi_{zz}\}(0,0,0). \tag{4.31}$$

In general, the residues of $M[f;s]$ at negative half-integers are continuous linear functionals of the rapidly decreasing function φ, concentrated on the vertex of the cone, $x=y=z=0$; cf. [3, p.254, last two lines].

To calculate the residue of $M[f;s]$ at negative integers, say $s=-p$, we write

$$\Phi(s,u) = \frac{\Phi_p(u)}{s+p} + \Phi_p(s,u), \tag{4.32}$$

where $\Phi_p(s,u)$ is analytic at $s=-p$. From (4.16) and (4.14), we have

$$\Phi_p(u) = \frac{(-1)^p}{4} a_p(u) = \frac{(-1)^p}{4(p!)} \left[\frac{\partial^p}{\partial t^p}\psi_1(u,ut)\right]\bigg|_{t=1}. \tag{4.33}$$

Since the expansion (4.22) can be differentiated any number of times with respect to t, $\Phi_p(u)$ has the asymptotic expansion

$$\Phi_p(u) = \frac{(-1)^p}{4(p!)} \sum_{l=0}^{N} \alpha_l^{(p)}(1)u^{l/2} + \mathrm{O}\{u^{(N+1)/2}\} \quad \text{as } u\to 0^+ \tag{4.34}$$

for any integer $N\geqslant 0$, where $\alpha_l(t)$ is given in (4.23). Furthermore, since the degree of the polynomial $\alpha_l(t)$ is $\frac{1}{2}(l-1)$ when l is odd, it follows from (4.23) that

$$\alpha_{2p-1}^{(p)}(t) = 0,$$

i.e., the term $u^{p-1/2}$ is absent from the expansion (4.34). For $s>-\frac{1}{2}$, (4.32) gives

$$M[f;s] = \frac{1}{s+p}\int_0^\infty u^{s-1/2}\Phi_p(u)\,du + \int_0^\infty u^{s-1/2}\Phi_p(s,u)\,du. \tag{4.35}$$

Each of the two integrals on the right-hand side defines an analytic function in Re $s>-\frac{1}{2}$. By the same analytic continuation argument used for $\Phi(s,u)$ and $M[f;s]$ above, it can be shown with the aid of (4.34) that these functions can also be extended to meromorphic functions in the entire plane. Since the expansion (4.34) does not contain the term $u^{p-1/2}$, $s=-p$ is not a pole of the first integral on the right of (4.35). (This agrees with our earlier remark that $M[f;s]$ has only simple poles at the negative integers.) Also because of this, it follows from (4.32) that the term $\beta_{2p-1}(s)u^{p-1/2}$ in the expansion (4.24) of $\Phi(s, u)$ must be in the series expansion of $\Phi_p(s, u)$. Consequently we may write

$$\Phi_p(s, u) = \beta_{2p-1}(s)u^{p-1/2} + A_p(s, u), \tag{4.36}$$

where $A_p(s, u)$ is analytic for s near $-p$ and has an asymptotic expansion of the form

$$A_p(s, u) \sim \sum_{l=0}^{\infty}{}' \gamma_l(s)u^{l/2} \quad \text{as } u \to 0^+. \tag{4.37}$$

The summation sign Σ' in (4.37) excludes the term $l = 2p - 1$. For Re $s > -\frac{1}{2}$, we have

$$\int_0^1 u^{s-1/2}\Phi_p(s, u)\,\mathrm{d}u = \beta_{2p-1}(s)\frac{1}{s+p} + \int_0^1 u^{s-1/2}A_p(s, u)\,\mathrm{d}u. \tag{4.38}$$

By analytic continuation, this equation holds also in the rest of the domain of analyticity, and so does

$$\int_0^{\infty} u^{s-1/2}\Phi_p(s, u)\,\mathrm{d}u = \beta_{2p-1}(s)\frac{1}{s+p} + \int_0^1 u^{s-1/2}A_p(s, u)\,\mathrm{d}u + \int_1^{\infty} u^{s-1/2}\Phi_p(s, u)\,\mathrm{d}u. \tag{4.39}$$

Since $\Phi_p(s, u)$ is analytic near $s = -p$ (see (4.32)), the two integrals on the right of (4.39) are analytic near $s = -p$. Coupling (4.35) and (4.39), we obtain

$$\operatorname*{Res}_{s=-p} M[f;s] = \left(\int_0^{\infty} u^{s-1/2}\Phi_p(u)\,\mathrm{d}u\right)\bigg|_{s=-p} + \beta_{2p-1}(s)\big|_{s=-p}. \tag{4.40}$$

The integral in (4.40) denotes not only its value when the integral exists but also its analytic continuation (see [9, p.153, Theorem 5]).

When $p = 0$, the sum in (4.25) is empty and we have $\beta_{-1}(s) \equiv 0$. Hence a combination of (4.40), (4.33) and (4.18) gives

$$\operatorname*{Res}_{s=0} M[f;s] = \tfrac{1}{2}\int_0^{\infty}\int_0^{2\pi} \varphi(z\cos\theta, z\sin\theta, z)\,\mathrm{d}\theta\,\mathrm{d}z. \tag{4.41}$$

(Here a change of variable $z = u^{1/2}$ has been made.) For $p = 1$, we have from (4.25)

$$\beta_1(-1) = -\tfrac{1}{2}\pi\varphi_z(0, 0, 0), \tag{4.42}$$

and from (4.18) and (4.33)

$$\Phi_1(u) = -\tfrac{1}{8}\sqrt{u}\int_0^{2\pi}\left[\varphi_x(\sqrt{u}\cos\theta, \sqrt{u}\sin\theta, \sqrt{u})\cos\theta + \varphi_y(\sqrt{u}\cos, \sqrt{u}\sin\theta, \sqrt{u})\sin\theta\right]\mathrm{d}\theta.$$

Since the integrals of $\cos\theta$ and $\sin\theta$ over the interval $[0, 2\pi]$ are zero, $\Phi_1(u) = O(u)$ near $u = 0$ and the integral

$$\int_0^{\infty} u^{-1-1/2}\Phi_1(u)\,\mathrm{d}u = -\tfrac{1}{8}\int_0^{\infty} u^{-1}\int_0^{2\pi}\left[\varphi_x(\sqrt{u}\cos\theta, \sqrt{u}\sin\theta, \sqrt{u})\cos\theta + \varphi_y(\sqrt{u}\cos\theta, \sqrt{u}\sin\theta, \sqrt{u})\sin\theta\right]\mathrm{d}\theta\,\mathrm{d}u$$

exists. Furthermore, since the θ-integral is $O(\sqrt{u})$ near $u=0$, after a change of variable $z=u^{1/2}$, integration by parts gives

$$\int_0^\infty u^{-1-1/2}\Phi_1(u)\,\mathrm{d}u = \tfrac{1}{4}\int_0^\infty \log z \int_0^{2\pi} \frac{\partial}{\partial z}\Big[\varphi_x(z\cos\theta, z\sin\theta, z)\cos\theta + \varphi_y(z\cos\theta, z\sin\theta, z)\sin\theta\Big]\,\mathrm{d}\theta\,\mathrm{d}z. \tag{4.43}$$

Inserting (4.42) and (4.43) in (4.40), we obtain

$$\operatorname*{Res}_{s=-1} M[f;s] = -\tfrac{1}{2}\pi\varphi_z(0,0,0) + \tfrac{1}{4}\int_0^\infty \log z \int_0^{2\pi} \frac{\partial}{\partial z}\Big[\varphi_x(z\cos\theta, z\sin\theta, z)\cos\theta + \varphi_y(z\cos\theta, z\sin\theta, z)\sin\theta\Big]\,\mathrm{d}\theta\,\mathrm{d}z. \tag{4.44}$$

Summarizing, we have the following lemma.

Lemma 2. *The Mellin transform $M[f;s]$ defined in (4.4) can be analytically continued to a meromorphic function in the entire s-plane with simple poles at*

$$s=0,\ -\tfrac{1}{2},\ -1,\ -\tfrac{3}{2},\ -2,\ -\tfrac{5}{2},\ \ldots .$$

The residue of $M[f;s]$ at $s=-p-\frac{1}{2}$ is a functional concentrated on the vertex of the cone $z^2=x^2+y^2$, $z\geqslant 0$, whereas the residue of $M[f;s]$ at $s=-p$ is a sum of two functionals, one concentrated on the surface of the cone and the other on its vertex.

There is a slight discrepancy between the above result and that given in [3, p.255], where the second term in (4.40) is missed out in the calculation.

Returning to (4.4), we now apply the Mellin inversion formula [9, p.151] and obtain

$$f(t) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} t^{-s} M[f;s]\,\mathrm{d}s, \tag{4.45}$$

where c is a positive number. To justify this, we make use of (4.16). From (4.14), it is evident that each $a_k(u)$ is a C^∞-function on $(0,\infty)$ and of rapid decrease at infinity. Note that $a_k(u)$ is actually a function of $\sqrt{u}$. Hence differentiation of this function introduces singularities at the origin. However, the integrals produced by successive integrations by parts remain convergent, and consequently

$$\int_0^\infty u^{s-1/2} a_k(u)\,\mathrm{d}u = O(s^{-p}) \quad \text{as } \operatorname{Im} s \to \pm\infty, \tag{4.46}$$

for $k=0,1,2,\ldots$ and for all $p\geqslant 1$. From (4.13), we have

$$R_N(u,0) = \psi_1(u,0) - \sum_{k=0}^{N}(-1)^k a_k(u)$$

from which it follows that $R_N(u,0)$ is also a C^∞-function of rapid decrease. Again by integration by parts,

$$\int_0^1 (1-t)^{s-1} R_N(u,t)\,\mathrm{d}t = \frac{1}{s}R_N(u,0) + \frac{1}{s}\int_0^1 (1-t)^s \frac{\partial}{\partial t} R_N(u,t)\,\mathrm{d}t. \tag{4.47}$$

(The last integral is convergent at $t=0$; cf (4.13)). From (4.13) and (4.15),

$$\frac{\partial}{\partial t}R_N(u,t)=\mathrm{O}\{(1-t)^N\} \quad \text{as } t\to 1^-.$$

Hence the last term in (4.47) is analytic for $\operatorname{Re} s>-N-1$, and is a C^∞-function of rapid decrease in u. Therefore we may rewrite (4.47) as

$$\int_0^1 (1-t)^{s-1}R_N(u,t)\,\mathrm{d}t=\frac{1}{s}E_N(u,s), \tag{4.48}$$

where $E_N(u,s)$ is a rapidly decreasing C^∞-function in u and is analytic in s for $\operatorname{Re} s>-N-1$. This, coupled with (4.16), gives

$$\Phi(s,u)=\sum_{k=0}^{N}\frac{(-1)^k}{4}\frac{a_k(u)}{s+k}+\frac{1}{s}E_N(u,s). \tag{4.49}$$

By an argument similar to that for (4.46),

$$\int_0^\infty u^{s-1/2}E_N(u,s)\,\mathrm{d}u=\mathrm{O}(s^{-p}) \quad \text{as } \operatorname{Im} s\to\pm\infty, \tag{4.50}$$

for all $p\geqslant 1$. Substituting (4.49) in (4.12), we obtain from (4.46) and (4.50) $M[f;s]=\mathrm{O}(s^{-p})$, as $\operatorname{Im} s\to\pm\infty$, for all $p\geqslant 1$. This not only justifies (4.45), but also allows the vertical line of integration $\operatorname{Re} s=c$ to be shifted to $\operatorname{Re} s=-N-\frac{1}{2}-\delta$ for any $0<\delta<\frac{1}{2}$. By Lemma 2 and the Cauchy residue theorem,

$$f(t)=\sum_{l=0}^{2N+1}a_l t^{l/2}+\mathrm{O}(t^{N+1/2+\delta}) \quad \text{as } t\to 0^+, \tag{4.51}$$

where

$$a_l=\operatorname*{Res}_{s=-l/2}M[f;s]. \tag{4.52}$$

The asymptotic expansion of the triple integral $T(\lambda)$ in (1.6) is now obtained from (4.1) and (2.31) with $x=\lambda$, $\alpha_l=\frac{1}{2}l$ and the polynomial $P_l(w)$ being the constant a_l. Consequently, we have

$$T(\lambda)\sim\sum_{l=0}^{\infty}a_l 2^{l/2}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{4}l)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{4}l)}\lambda^{-l/2-1}, \tag{4.53}$$

as $\lambda\to+\infty$.

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